- There are 4 hours available for the problems.
- Every problem is worth at most 10 points.

MOAWOA Solutions
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Problem 1.
Let n be a natural number and define A to be the $n \times n$ matrix with $A_{i,i+1} = A_{i+1,i} = i+1$ and $A_{ii} = i^2+1$ whenever $1 \leq i < n$, $A_{nn} = n^2$ and all other entries are zero. Calculate $\det A$.

Solution 1. We will prove by induction that $\det A = n!^2$. This is trivial for $n=1$ and $n=2$, now suppose it is true for some $n \geq 2$. Then we get

$$\begin{vmatrix} 2 & 2 & \dots & 0 & 0 \\ 2 & 5 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & n^2+1 & n+1 \\ 0 & 0 & \dots & n+1 & (n+1)^2 \end{vmatrix} = \begin{vmatrix} 2 & 2 & \dots & \dots & 0 \\ 2 & 5 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & n^2 & 0 \\ 0 & \dots & \dots & n+1 & (n+1)^2 \end{vmatrix}$$

$$= (n+1)^2 \begin{vmatrix} 2 & 2 & \dots & \dots & \dots \\ 2 & 5 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & n & n^2 \end{vmatrix} - (n+1) \begin{vmatrix} 2 & 2 & \dots & \dots & 0 \\ 2 & 5 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & n & 0 \end{vmatrix}$$

$$= (n+1)^2 n!^2 = (n+1)!^2.$$

Solution 2. Denote by B the matrix with diagonal entries $1, 2, \dots, n$, superdiagonal entries equal to 1 and all other entries equal to 0. Then $A = BB^t$, thus $\det A = (\det B)(\det B^t) = n!^2$.

Problem 2.
Find all permutations σ on the set $\{1, 2, \dots, n\}$ satisfying

$$\sum_{i=1}^{n} \frac{\sigma(i)}{i} = n.$$

Solution 1. We will prove by induction that any σ, not equal to the identity, satisfies

$$\sum_{i=1}^{n} \frac{\sigma(i)}{i} > n.$$

This would then imply that the only permutation that is a solution is the identity. For $n = 2$ this is evident, since $2+\frac{1}{2} > 2$. Now suppose the statement is true for some $n \geq 2$. Let σ be a permutation on the set $\{1, 2, \ldots, n+1\}$. If $\sigma(n+1) = n+1$ then clearly we are done, so let us suppose $\sigma(n+1) \neq n+1$. Then we find $i_0, j_0 \in \{1, 2, \ldots, n\}$ such that $\sigma(i_0) = n + 1$ and $\sigma(n + 1) = j_0$. Now let us define a permutation σ_0 on $\{1, 2, \ldots, n\}$ by $\sigma_0(i) = \sigma(i)$ when $i \neq i_0$ and $\sigma(i_0) = j_0$. We get

$$\begin{aligned}\sum_{i=1}^{n+1} \frac{\sigma(i)}{i} &= \frac{n+1}{i_0} + \frac{j_0}{n+1} - \frac{j_0}{i_0} + \sum_{i=1}^{n} \frac{\sigma_0(i)}{i} \\ &\geq \frac{n+1-j_0}{i_0} + \frac{j_0}{n+1} + n > \frac{n+1-j_0}{n+1} + \frac{j_0}{n+1} + n = n+1.\end{aligned}$$

Solution 2. By the Arithmetic Mean-Geometric Mean Inequality we have

$$\sum_{i=1}^{n} \frac{\sigma(i)}{i} \geq n \sqrt[n]{\frac{\sigma(1)}{1} \frac{\sigma(2)}{2} \cdots \frac{\sigma(n)}{n}} = n$$

and we have equality if and only if $\frac{\sigma(1)}{1} = \frac{\sigma(2)}{2} = \ldots = \frac{\sigma(2)}{2}$. Because $\frac{\sigma(n)}{n} \leq 1 \leq \frac{\sigma(1)}{1}$ this only happens when $\frac{\sigma(i)}{i} = 1$ for all $1 \leq i \leq n$, i.e. σ must be the identity.

Problem 3.
Let n be a natural number. Give (explicitly) real numbers $a_0, a_1, \ldots, a_n$, not all equal to zero, such that for all n times continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ the following equation holds

$$\sum_{k=0}^{n} a_k f(kx) = \mathcal{O}(x^n) \text{ for } x \text{ small enough.}$$

Solution. Take $a_k = \binom{n}{k}(-1)^{n-k}$ and let $g(x) = \sum a_k f(kx)$. First we notice

$$\sum_{k=0}^{n} k^m \binom{n}{k} (-1)^{n-k} = \frac{d^m}{dx^m}(e^x - 1)^n|_{x=0} = \begin{cases} 0 & \text{if } 0 \leq m < n \\ n! & \text{if } m = n \end{cases}$$

This shows that $g^{(n)}(0) = n!f^{(n)}(0)$ and $g(0) = g'(0) = \ldots = g^{(n-1)}(0) = 0$. Hence we may apply l'Hôpitals theorem successively to obtain that

$$\lim_{x \to 0} \frac{g(x)}{x^n} = \frac{g^{(n)}(0)}{n!} = f^{(n)}(0),$$

which proves that $g(x) = \mathcal{O}(x^n)$ for x small enough.

Remark. Denote by T_x the operator $(T_x f)(x_0) = f(x_0 + x) - f(x_0)$. Intuitively one might suspect that $x^{-n}(T_x^n f)(0)$ has limit $f^{(n)}(0)$ as $x \to 0$, indeed this is true. One easily shows by induction that $(T_x^n f)(0) = g(x)$.

Problem 4.

For any natural number n let $\pi(n)$ be the number of sets of natural numbers whose elements add up to n and let $\pi_2(n)$ be the number of these sets that contain at least one power of 2. Prove that $\pi_2(n+1) = \pi(n)$. ***Remark:*** *in this problem* 1 *is considered to be a power of* 2.

Solution 1. Denote by $\Pi(n)$ the collection of sets of natural numbers whose elements add up to n and let $\Pi_2(n)$ be the collection of sets in $\Pi(n)$ that contain at least one power of 2. Consider the map $f : \Pi_2(n+1) \to \Pi(n)$ that sends any set $A \in \Pi_2(n+1)$ to $(A \setminus \{2^k\}) \cup \{2^l | 0 \leq l < k\}$, where 2^k is the smallest power of 2 in A. One easily checks that f is a bijective function and thus $\pi_2(n+1) = \pi(n)$.

Solution 2. Let us prove this statement by induction. The statement is clearly true for $n = 1$. Now suppose that $\pi_2(k+1) = \pi(k)$ for all $1 \leq k < n$. Notice that every subset of $\mathbb{N}$ can be written as a union of a subset that contains only powers of two and a subset that doesnt contain any power of two. The amount of ways to write a natural number $n + 1 - k$ as a sum of powers of two is one. Thus we must conclude that

$$\pi_2(n+1) = 1 + \sum_{k=1}^{n} (\pi(k) - \pi_2(k)) \cdot 1 = 1 + \pi(n) - \pi_2(1) = \pi(n).$$

Solution 3. Define $\pi(0) = 1$ for convenience. Clearly $\pi(n) \leq 2^n$ and thus the series $\sum \pi(n)x^n$ converges for $|x| < \frac{1}{2}$. Now let $|x| < \frac{1}{2}$, notice that

$$\lim_{N\to\infty} \left| \prod_{n=1}^{N} (1 + x^n) - \sum_{n=0}^{N} \pi(n)x^n \right| \leq \lim_{N\to\infty} \sum_{n=N+1}^{\infty} \pi(n)|x|^n = 0$$

and since every natural number has a unique binary expansion

$$\lim_{N\to\infty} \prod_{j=0}^{N-1} (1 + x^{2^j}) = \lim_{N\to\infty} \sum_{n=0}^{2^N-1} x^n = \frac{1}{1-x}.$$

It follows that $\pi_2(n+1) = \pi(n)$, because for all $|x| < \frac{1}{2}$

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} (\pi(n) - \pi_2(n))x^n &= (1-x) \prod_{n=1}^{\infty} (1 + x^n) = (1-x) \sum_{n=0}^{\infty} \pi(n)x^n \\ &= 1 + \sum_{n=1}^{\infty} (\pi(n) - \pi(n-1))x^n. \end{aligned}$$

Exercise 5.

Find all natural numbers n such that for every positive divisor d of n we have $n \mid d^2$ or $d^2 \mid n + k$ for some positive divisor k of n.

Solution. Let p be prime. For any two powers e and e' of p we have that $e' \mid e$ or $e \mid e'$. So if n is a prime power then $n \mid d^2$ or $d^2 \mid n$. The latter implies $d^2 \mid n + n$. So all prime powers, including 1, satisfy the condition.

Now suppose n has more than one prime divisor. Let p and q be different primes dividing n and let l and m be the unique natural numbers satisfying $p^l \mid n$, $p^{l+1} \nmid n$, $q^m \mid n$ and $q^{m+1} \nmid n$. Now look at the divisor $d = \frac{n}{p^l}$ of n. Clearly $n \nmid d^2$ so there must exist a positive divisor k of n such that $d^2 \mid n + k$. By $d \mid n$ we find that $d \mid k$ and $k \mid n$ so $k = \frac{n}{p^s}$ for some non-negative integer $s \leq l$. This yields $q^{2m} \mid n + \frac{n}{p^s}$ and hence $q^m \mid p^s + 1$. Similarly we find a non-negative $t \leq m$ such that $p^l \mid q^t + 1$. This gives the estimation $p^l \leq q^t + 1 \leq q^m + 1 \leq p^s + 2$. So $p = 2$, $p = 3$ or $l = s$. If $p = 2$ and $s < l$ we have $(l, s) = (2, 1)$ or $(1, 0)$. So we find respectively $q^m \mid 3$ and $q^m \mid 2$. The second gives a contradiction with $p \neq q$, the first one gives $q = 3$ and $m = 1$. If $p = 3$ and $s < l$ we must have $l = 1$ and $s = 0$. By the same constraints on q we find for $t < m$ the same cases, so the possible solutions are 6 and 12.

Now we can assume $l = s$ and $m = t$. Writing $A = p^l$ and $B = q^m$ gives $B \mid A + 1$ and $A \mid B + 1$. The case $A = B$ is clearly impossible so assume without loss of generality that $A < B$. Then we get $A \leq \frac{B+1}{2} \leq \frac{A+2}{2} = \frac{A}{2} + 1$ so $A \leq 2$. Clearly $A \neq 1$ so $A = 2$ and we find $B = 3$.

Note that we have proven that for any two distinct prime divisors of n one equals 2 and the other equals 3. Hence n has at most two prime divisors and the discussion above shows that 6 and 12 are the only possible solutions with more than one prime divisor. A quick check shows they indeed satisfy the conditions of the exercise.

So all such numbers are 6, 12 and all prime powers, including 1.

Exercise 6.

Let H and K be subgroups of a finite group G. Suppose that $gH \cap Kg$ consists of one element for all $g \in G$. Prove that $|H| \cdot |K|$ divides $|G|$.

Solution 1. We will imitate the proof of Lagrange's theorem and define an equivalence relation on G such that the size of each equivalence classes is equal to $|H| \cdot |K|$.

Define $g \sim g'$ if and only if there exists $h \in H$ and $k \in K$ such that $kgh = g'$. We have $g \sim g$ since $e_G \in H \cap K$ and if $g \sim g'$ then from $kgh = g'$ we get $k^{-1}g'h^{-1} = g$ so $g' \sim g$. Now if $g \sim g'$ and $g' \sim g''$ we can find $h, h' \in H$ and $k, k' \in K$ such that $kgh = g'$ and $k'g'h' = g''$. Hence we have $(k'k)g(hh') = g''$. So $\sim$ defines a equivalence relation on G and the equivalence class of g clearly equals KgH. The map $H \times K \to HgK$, $(h, k) \mapsto kgh$ is clearly onto. It is one-to-one since from $kgh = k'gh'$ we get $(k'^{-1}k)g = g(h'h^{-1})$. By the assumption we now get $k'^{-1}k = e_G = h'h^{-1}$ so $(h, k) = (h', k')$. So the size of each equivalence class is $|H| \cdot |K| = |H \times K|$ and the result follows as they partition out G.

Solution 2. Let $(h, k) \in H \times K$ act on G by sending $g \in G$ to kgh^{-1}. By the orbit counting formula we have that the number of orbits equals

$$\frac{1}{|H \times K|} \sum_{(h,k) \in H \times K} |\mathrm{Fix}(h, k)|$$

where $\mathrm{Fix}(h, k) = \{g \in G \mid kgh^{-1} = g\}$. If $\mathrm{Fix}(h, k)$ is not empty we use the assumption to conclude as in solution 1 that $h = k = e_G$. Hence $\sum_{(h,k) \in H \times K} |\mathrm{Fix}(h, k)| = |\mathrm{Fix}(e_G, e_G)| = |G|$.