# MOAWOA
## April 12, 2013

**Problem 1.** Let $n \in \mathbb{N}$ be odd and let $\zeta = e^{2\pi i/n}$. Calculate

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\zeta^i + \zeta^j}.$$

**Solution 1.** We notice that $\{\zeta^k | 0 \leq k \leq n-1\} = \{\zeta^k | 1 \leq k \leq n\}$, therefore

$$\zeta \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\zeta^i + \zeta^j} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\zeta^{i-1} + \zeta^{j-1}} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \frac{1}{\zeta^i + \zeta^j} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\zeta^i + \zeta^j}.$$

Thus the sum equals 0 when $n > 1$, because then $\zeta \neq 1$. For $n = 1$ the sum equals $\frac{1}{2}$.

**Solution 2.** We notice that $\{\zeta^k | 1 - i \leq k \leq n - 1 - i\} = \{\zeta^k | 1 \leq k \leq n\}$ for any $i \in \mathbb{Z}$, therefore

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\zeta^i + \zeta^j} = \sum_{i=1}^{n} \zeta^{-i} \sum_{j=1}^{n} \frac{1}{1 + \zeta^{j-i}} = \sum_{i=1}^{n} \zeta^{-i} \sum_{j=1}^{n} \frac{1}{1 + \zeta^j} = \frac{\zeta^{-n} - 1}{\zeta^{-1} - 1} \sum_{j=1}^{n} \frac{1}{1 + \zeta^j} = 0$$

for $n > 1$ since $\zeta^{-1}$ is an $n^{th}$ root of unity.

**Problem 2.** Let $\mathcal{A}$ be a countable collection of sets. Does it follow that there exists a countable collection $\mathcal{B}$ of pairwise disjoint sets such that every set in $\mathcal{A}$ is a finite union of sets in $\mathcal{B}$?

**Solution.** No. Consider the collection $\mathcal{A} = \{A_1, A_2, \ldots\}$ where $A_n = \mathbb{N} \setminus \{n\}$. Suppose that a collection $\mathcal{B} = \{B_1, B_2, \ldots\}$ with the desired properties exists. It is enough to prove that every set $B_n$ must consist of only one element. Assume that there exist two distinct natural numbers $i$ and $j$ such that $i, j \in B_n$. As the sets in $\mathcal{B}$ are pairwise disjoint and $i \in A_j$ we must have $B_n \subset A_j$. But then $j \in \mathbb{N} \setminus \{j\}$ and we have reached a contradiction.

**Problem 3. (by Sjoerd Boersma)** Alex and Bo play a game:
Alternatingly they choose a real non-zero number. This real number cannot be a number that was already chosen, and not an additive inverse of a number that has already been chosen. A player wins the game when he chooses a number such that the sum of all chosen numbers equals 0. The game ends when $10^{100}$ numbers have been chosen and no one has won yet. Suppose Alex may start. If both players play an optimal strategy, what will their game result in?

**Solution.** It will result in a draw. It is always possible to choose a number that has a larger absolute value than all the previously chosen numbers (and hence may be chosen), and such that the total sum equals one of the previously chosen numbers or its additive inverse (namely the one with the largest absolute value). Consequently each player can make sure that the other player doesn't win. Concretely:
Suppose $a_1, a_2, \ldots, a_n$ have been chosen and without loss of generality $|a_1| < |a_2| < \ldots < |a_n|$. Let $a = a_1 + a_2 + \ldots + a_n$. Now let $b = -a - \operatorname{sgn}(a)|a_n|$, where sgn denotes the sign of $a$. Then $|b| = |a| + |a_n| > |a_n| > |a_i|$ for all $i = 1, 2, \ldots, n$ and $|a + b| = |a_n|$.

**Problem 4.** Calculate

$$\sum_{n=1}^{\infty} \frac{1}{n(9n^2-1)}.$$

**Solution 1.** First we notice that

$$\sum_{n=1}^{\infty} \frac{1}{n(9n^2-1)} = \frac{3}{2} \sum_{n=1}^{\infty} \left( \frac{1}{3n-1} + \frac{1}{3n} + \frac{1}{3n+1} - \frac{1}{n} \right) = -\frac{3}{2} + \lim_{N\to\infty} \frac{3}{2} \sum_{n=N+1}^{3N+1} \frac{1}{n}.$$

We have

$$\log\left(3 - \frac{1}{N+1}\right) = \int_{N+1}^{3N+2} \frac{dx}{x} \leq \sum_{n=N+1}^{3N+1} \frac{1}{n} \leq \int_{N}^{3N+1} \frac{dx}{x} = \log\left(3 + \frac{1}{N}\right).$$

Taking the limit $N \to \infty$ yields

$$\sum_{n=1}^{\infty} \frac{1}{n(9n^2-1)} = \frac{-3 + 3\log 3}{2}.$$

**Solution 2.** For $|x| \leq 1$ define

$$f(x) = \sum_{n=1}^{\infty} \frac{x^{3n+1}}{n(9n^2-1)}.$$

We notice that, for $|x| < 1$ we have

$$f^{(3)}(x) = 3 \sum_{n=1}^{\infty} x^{3n-2} = \frac{3x}{1-x^3}.$$

Now let $\alpha = e^{2\pi i/3}$. We see that

$$\frac{3x}{1-x^3} = -x \left( \frac{1}{x-1} + \frac{\alpha}{x-\alpha} + \frac{\alpha^2}{x-\alpha^2} \right) = \frac{1}{1-x} + \frac{\alpha^2}{\alpha - x} + \frac{\alpha}{\alpha^2 - x}.$$

Hence for $|x| < 1$

$$f^{(2)}(x) = f^{(2)}(x) - f^{(2)}(0) = \int_0^x \sum_{k=0}^{2} \frac{\alpha^{-k}}{\alpha^k - y} dy = \frac{\pi}{\sqrt{3}} - \sum_{k=0}^{2} \alpha^{-k} \log(\alpha^k - x)$$

$$f'(x) = \frac{\pi}{\sqrt{3}} x - \int_0^x \sum_{k=0}^{2} \alpha^{-k} \log(\alpha^k - y) dy = \frac{\pi}{\sqrt{3}} x - \sum_{k=0}^{2} \alpha^{-k} (x - \alpha^k) \log(\alpha^k - x)$$

Thus we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n(9n^2-1)} = f(1) = \lim_{x \to 1} f(x) &= \frac{\pi}{2\sqrt{3}} - \int_0^1 \sum_{k=0}^{2} \alpha^{-k} (y - \alpha^k) \log(\alpha^k - y) dy \\
&= \frac{\pi}{2\sqrt{3}} - \left[ \sum_{k=0}^{2} \alpha^{-k} \frac{(y-\alpha^k)^2}{2} \left( \log(\alpha^k - y) - \frac{1}{2} \right) \right]_0^1 \\
&= -2\mathrm{Re} \left( \alpha \frac{(-i\sqrt{3}\alpha)^2}{2} \left( \log(i\sqrt{3}\alpha) - \frac{1}{2} \right) \right) = \frac{-3 + 3\log 3}{2}
\end{aligned}$$

where we have used the identities $1 + \alpha + \alpha^2 = 0$ and $1 - \alpha^2 = -i\sqrt{3}\alpha$.

**Solution 3.** We notice that $\alpha^n + \alpha^{2n} = 2$ if $3|n$ and $-1$ otherwise. Thus

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n(9n^2-1)} &= \frac{3}{2} \sum_{n=1}^{\infty} \left( \frac{1}{3n+1} + \frac{1}{3n-1} - \frac{2}{3n} \right) \\
&= \frac{3}{2} \left( -1 + \log(1-\alpha) + \log(1-\alpha^2) \right) = \frac{3}{2} \left( -1 + 2\log|1-\alpha| \right) = \frac{-3 + 3\log 3}{2}.
\end{aligned}$$

**Problem 5.** Let $M$ be an invertible complex square matrix. Prove that

$$\left| \det \left( \frac{M^{-1} + M^\dagger}{2} \right) \right| \geq 1.$$

For what matrices $M$ do we have equality?

**Solution.** Evidently $M^\dagger M$ is hermitian and thus it can be diagonalized by a (unitary) matrix $U$ and a diagonal matrix $D$ with Eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ on it's diagonal. Now let $v_i$ be an Eigenvector of $M^\dagger M$ corresponding to Eigenvalue $\lambda_i$. Then $\lambda_i v^\dagger v = v^\dagger (M^\dagger M v) = (Mv)^\dagger (Mv)$. Since $v^\dagger v > 0$ and $(Mv)^\dagger (Mv) \geq 0$ we must conclude that $\lambda_i$ is real and non-negative. Using this result we get

$$\begin{aligned}
\det \left( \frac{\mathbb{I} + M^\dagger M}{2} \right) &= \det \left( U^{-1} \frac{\mathbb{I} + D}{2} U \right) = \det \left( \frac{\mathbb{I} + D}{2} \right) \\
&= \prod_{i=1}^{n} \frac{1 + \lambda_i}{2} \geq \prod_{i=1}^{n} \sqrt{\lambda_i} = \sqrt{\det (M^\dagger M)} \\
&= \sqrt{\det M^\dagger \det M} = |\det M|.
\end{aligned}$$

We conclude that

$$\left|\det\left(\frac{M^{-1}+M^\dagger}{2}\right)\right| = |\det M|^{-1}\det\left(\frac{\mathbb{I}+M^\dagger M}{2}\right) \geq 1.$$

Obviously equality happens exactly when all $\lambda_i$ are 1, i.e. when $M^\dagger M = U^{-1}\mathbb{I}U = \mathbb{I}$. We conclude that we have equality if and only if $M$ is unitary.

**Problem 6.** Does there exist a polynomial $p$ of degree 24 with coefficients equal to $\pm 1$ such that $|p(z)| \leq 9$ on the (complex) unit circle?

**Solution.** Consider the polynomial $q(z) = 1 - z + z^2 + z^3 + z^4$. We notice that

$$\begin{aligned}|q(e^{i\theta})|^2 &= |e^{2i\theta}(2\cos(2\theta) + 2i\sin(\theta) + 1)|^2 = 4\cos^2(2\theta) + 4\cos(2\theta) + 1 + 4\sin^2(\theta)\\ &= 3 + 2\cos(2\theta)(2\cos(2\theta)+1) \leq 9.\end{aligned}$$

We conclude that the polynomial $p(z) = q(z)q(z^5)$ satisfies the desired properties.

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